

9/17 Recitation

1. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$. Determine $N(A)$ by finding vectors that span $N(A)$. What is $N(A)$, geometrically? Where does it live?

First, we put A in RREF: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence the only solution to $A\mathbf{x} = \mathbf{0}$ is $x_1 = x_2 = x_3 = 0$ (or $\mathbf{x} = \mathbf{0}$). Hence $N(A) = \{\mathbf{0}\} \subset \mathbb{R}^3$, which is the span of $\mathbf{0} \in \mathbb{R}^3$.

2. Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$. Consider the question: Is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $C(A)$? Let's solve this in two ways.

(a) Solve using the interactive graph at <https://www.geogebra.org/m/btwdyx3f>.

The graph depicts $C(A)$, which is a plane in $\mathbb{R}^3$. Since $\mathbf{b}$ is not in the plane, it is not in $C(A)$.

(b) Solve by solving a related system of equations.

To show the vector is not in $C(A)$, we need to show that it cannot be expressed as a linear combination of the columns of A ; in other words, the matrix equation $A\mathbf{x} = \mathbf{b}$. Row reducing the augmented matrix gives us

$$[A \mid \mathbf{b}] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Since there is a pivot in the final column, there is no solution; hence $\mathbf{b} \notin C(A)$.

3. (a) Show that if an $m \times n$ matrix A has a pivot in every row, then $C(A) = \mathbb{R}^m$.

If there is a pivot in every row, then $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$ since there will never be a pivot in the final column of the augmented matrix $[A \mid \mathbf{b}]$. Since $A\mathbf{x}$ is a linear combination of the columns of A , this implies that every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A . In other words, every $\mathbf{b} \in \mathbb{R}^m$ is in the span of the columns of A , i.e. $C(A)$. Hence $C(A)$ is all of $\mathbb{R}^m$.

- (b) In class, we argued geometrically that if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are not scalar multiples of each other, then $\text{Span}\{\mathbf{v}, \mathbf{w}\} = \mathbb{R}^2$ (i.e. the span is a plane). Prove this algebraically using part (a).

Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and consider the matrix $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$. By part (a), if A has a pivot in every row, then $\mathbf{v}$ and $\mathbf{w}$ span $\mathbb{R}^2$. Hence we must show that A has two pivots.

First note that if $v_1 = w_1 = 0$, then $\mathbf{v}$ is a multiple of $\mathbf{w}$ (or vice versa). If $v_2 \neq 0$, then $\mathbf{w} = \frac{w_2}{v_2}\mathbf{v}$; and if $w_2 \neq 0$, then $\mathbf{v} = \frac{v_2}{w_2}\mathbf{w}$.

Let's assume that $v_1 \neq 0$ (a similar argument holds if $w_1 \neq 0$). Note that $w_1 = \frac{w_1}{v_1}v_1$ (i.e. w_1 is a multiple of v_1). Hence if $\mathbf{v}$ was a multiple of $\mathbf{w}$, we would also have $w_2 = \frac{w_1}{v_1}v_2$. Since $\mathbf{v}$ is not a multiple of $\mathbf{w}$, we necessarily have that $w_2 \neq \frac{w_1}{v_1}v_2$.

Now let's look for the pivots of A . The row replacement: Replace R_2 with $R_2 - \frac{v_2}{v_1}R_1$ yields the matrix $A = \begin{bmatrix} v_1 & w_1 \\ 0 & w_2 - \frac{w_1 v_2}{v_1} \end{bmatrix}$. Since $w_2 \neq \frac{w_1}{v_1}v_2$, we have that A has a pivot in both rows. The result now follows from part (a).

- (c) **Looking Ahead:** The vectors $\mathbf{v}$ and $\mathbf{w}$ in part (b) are called **linearly independent**. We will see later this week that two $\mathbf{v}$ and $\mathbf{w}$ are linearly independent if and only if the matrix $\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ has a pivot in each column. Redo part (b), using this fact.

Since there is a pivot in every column and the matrix $\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ is square, it also has a pivot in every row. Hence by part (a), the span of the two vectors is $\mathbb{R}^2$.